In special relativity, many quantities may appear similar, e.g. with upper and lower indices, but they can play very different roles.

We can break them down into 3 categories:

i) Tensors: The kinematic and dynamical quantities we work with will (almost always) be represented by some (p,q)-tensor where p (q) is the number of upper (lower) indices. The distinction between upper and lower indices is exactly the distinction between a representation r of $SO(1,3)$ (upper) and a dual representation (lower).

Tensors can be combined with or without index contraction:

$$V_\mu T^{\mu\nu} = G^\nu$$
$$V_\mu T^{\lambda\nu} = H_\mu{}^{\lambda\nu}$$

Some tensors can be represented by matrices, e.g. $V^\mu = \begin{pmatrix}\vdots\end{pmatrix}$, $V_\mu = (\cdots)$, $T^{\mu\nu} = \begin{pmatrix}\vdots & \vdots & \vdots\end{pmatrix}$, $T_{\mu\nu} = \begin{pmatrix}\vdots & \vdots & \vdots\end{pmatrix}$, $T_\mu{}^\nu = \begin{pmatrix}\vdots & \vdots & \vdots\end{pmatrix}$

but many cannot, e.g. $H_{\mu\nu\lambda}$.

The indices on a tensor essentially tell us how it transforms:

Each upper index transforms like a vector, i.e. $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$

Each lower index transforms like a dual vector, i.e. $V_\mu \to V_{\mu'} = \Lambda^\mu{}_{\mu'} V_\mu$

ii) The metric tensor: The metric tensor $g_{\mu\nu}$ takes an element of r (a vector index) to an element of $\tilde{r}$ (a dual vector index). This is often called "lowering" the index, i.e. $g_{\mu\nu}V^\nu = V_\mu$

The inverse metric $g^{\mu\nu}$ does the opposite, $g^{\mu\nu}V_\nu = V^\mu$.

The metric is a true tensor and so transforms accordingly, i.e. $g_{\mu\nu} \to g_{\mu'\nu'} = \underbrace{\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} g_{\mu\nu}}_{\Lambda^{-1T} g \Lambda^{-1}}$

However we know: $\Lambda^T g \Lambda = g \Rightarrow g = \Lambda^{-1T} g \Lambda^{-1} \Rightarrow g = g'$

Hence, the Λ's are called "isometries" of the metric. They do not change its form.

Aside: Secretly $g_{\mu\nu}$ is also a dynamical field, but it takes general relativity to see that!

In special relativity we often use $\eta_{\mu\nu}$ for $g_{\mu\nu}$.

The components of $\eta_{\mu\nu} = \begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$ and happen also to be the components of $\eta^{\mu\nu} = \begin{pmatrix}-1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$

Then: $\eta_{\mu\nu}\eta^{\nu\lambda} = \delta_\mu{}^\lambda = \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$, $\eta_{\mu\nu}\eta^{\nu\mu} = 4$

Note: An arbitrary $(0,2)$-tensor may seem like it can lower indeces, e.g. $M_{\mu\nu}V^\nu$ but the result is not V_μ, it is a new tensor, i.e. $M_{\mu\nu}V^\nu = G_\mu$. Only the metric provides the unique map from a vector to its corresponding dual, $g_{\mu\nu}V^\nu = V_\mu$.

iii) Transformations: $\Lambda^{\mu'}{}_\mu$, $\Lambda_{\mu'}{}^\mu = (\Lambda^{\mu'}{}_\mu)^{-1T}$ these always operate on tensors, can always be represented by a matrix, always carry one index from the old coordinates and one from the new

We never transform transformations! That would be silly!

Index notation and matrices

Okay, for real let's look at an example:

We can passively rotate a 2D vector $dx^\mu = \begin{pmatrix} dx \\ dy \end{pmatrix}$ with a matrix: $\underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}}_{\text{the order here matters!}} = \begin{pmatrix} \cos\theta dx + \sin\theta dy \\ -\sin\theta dx + \cos\theta dy \end{pmatrix} = \begin{pmatrix} dx' \\ dy' \end{pmatrix}$

same thing!

Or we could say: $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \Lambda^{1'}{}_{1} = \cos\theta,\ \Lambda^{1'}{}_{2} = \sin\theta,\ \Lambda^{2'}{}_{1} = -\sin\theta,\ \Lambda^{2'}{}_{2} = \cos\theta$

then $dx^\mu \to dx^{\mu'} = \Lambda^{\mu'}{}_{\mu} dx^\mu = \Lambda^{\mu'}{}_{1} dx + \Lambda^{\mu'}{}_{2} dy$

or

$$\left.\begin{aligned} dx' &= \cos\theta\, dx + \sin\theta\, dy \\ dy' &= -\sin\theta\, dx + \cos\theta\, dy \end{aligned}\right\}$$

But we could also say: $dx^{\mu'} = \underbrace{dx^\mu \Lambda^{\mu'}{}_{\mu}} = dx \Lambda^{\mu'}{}_{1} + dy \Lambda^{\mu'}{}_{2}$

We end up with the same thing even though we switched order!

So one huge advantage to index notation is that we don't have to worry about order. But if we do want to use matrices (sometimes they are useful) we have to get the order right!

$dx^{\mu'} = \Lambda^{\mu'}{}_{\mu} dx^\mu = dx^\mu \Lambda^{\mu'}{}_{\mu} = \Lambda d\vec{x}$ but $\underline{\underline{\text{not}}}$ $d\vec{x}\Lambda$!

The trick is to always get repeated indices directly next to each other.

So: $M_{\mu\nu} T^{\lambda\mu} = T^{\lambda\mu} M_{\mu\nu} = TM$ but not MT!

Sometimes we need to reorder indices to get this to work. Some important things to note:

$\eta_{\mu\nu} = \eta_{\nu\mu}$, starting w/ $\Lambda \sim \Lambda^{\mu'}{}_{\mu} \Rightarrow$

$$\begin{aligned} \Lambda_{\mu}{}^{\mu'} &= (\Lambda^{\mu'}{}_{\mu})^T \\ \Lambda^{\mu}{}_{\mu'} &= (\Lambda^{\mu'}{}_{\mu})^{-1} \\ \Lambda_{\mu'}{}^{\mu} &= (\Lambda^{\mu'}{}_{\mu})^{-1\,T} \end{aligned}$$

When we actually calculate probabilities for comparison with experiment we work in a limit where things look more like particles. Recall that for particles in 3D the main degree of freedom is position $\vec{r}(t)$, and interesting kinematic quantities are obtained by taking derivatives with respect to time, e.g. $\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{p} = m\vec{v}$, $E = \frac{1}{2}mv^2$, $\vec{a} = \frac{d\vec{v}}{dt}$.

So a naive way to generalize this to 4D is: $d\vec{r} \to dx^\mu = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$, $\vec{v} \to \frac{dx^\mu}{dt}$, etc.

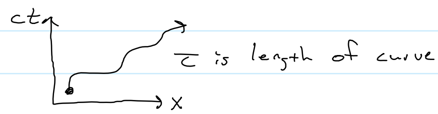
But we immediately encounter a problem.
An important question to ask is "Does our new object transform like a tensor?"
The answer is no!

$$\frac{dx^\mu}{dt} \longrightarrow \frac{dx^{\mu'}}{dt'} = \Lambda^{\mu'}{}_{\mu} \underbrace{\frac{dx^\mu}{dt'}}$$

this guy dt' does not transform like any tensor (scalar, vector, dual, etc.). In fact we know how it transforms: it transforms like one component of a vector, i.e. dx^μ!

Note: To transform like a <u>tensor</u> a quantity must <u>only</u> transform with "factors" of Λ.

To remedy this we need something that parameterizes the path of a particle that replaces time. One obvious solution is the "length" of the path.



Then for sub-luminal particles $\frac{d}{dt} \Rightarrow \frac{d}{d\tau}$ where $\tau = \int \sqrt{-ds^2}$
$= \int \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$

This is often called the "rest" time because in the rest frame of a particle $dx = dy = dz = 0 \Rightarrow d\tau = cdt$

It solves our problem since $d\tau$ is an invariant!

So... introducing the 4-velocity $U^\mu \equiv c\frac{dx^\mu}{d\tau}$ which is a true tensor, $U^\mu \to U^{\mu'} = \Lambda^{\mu'}{}_{\mu} U^\mu$!

This just gets the units right since $\frac{dx^\mu}{d\tau}$ is unitless!

Let's see what U^μ looks like in detail:

As seen in S: $U^0 = c^2 \frac{dt}{d\tau}$ But: $d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

$U^1 = c\frac{dx}{d\tau} = c\frac{dx}{dt}\frac{dt}{d\tau}$ $\Downarrow$

$U^2 = c\frac{dy}{d\tau} = c\frac{dy}{dt}\frac{dt}{d\tau}$ $\frac{d\tau^2}{dt^2} = c^2 - v^2$

$U^3 = c\frac{dz}{d\tau} = c\frac{dz}{dt}\frac{dt}{d\tau}$ $\Downarrow$

$\frac{dt}{d\tau} = \frac{\gamma}{c}$

$$\left.\begin{aligned} U^0 &= \gamma c \\ U^1 &= \gamma v_x \\ U^2 &= \gamma v_y \\ U^3 &= \gamma v_z \end{aligned}\right\} \quad U^\mu = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}$$

Note: The γ's utilize all components of $\vec{v}$ since $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

We can now define perhaps the most important quantity for studying $\underbrace{\text{collisions}}_{\text{includes decays a.k.a. explosions}}$, 4-momentum $P^\mu = mU^\mu$

invariant rest mass

Then immediately: $P^\mu = \begin{pmatrix} m\gamma c \\ m\gamma\vec{v} \end{pmatrix} \Rightarrow P^\mu_{rest} = \begin{pmatrix} mc \\ 0 \end{pmatrix}$

look like kinetic energy

To interpret this: For $\frac{v}{c} \ll 1$ $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \Rightarrow P^\mu = \begin{pmatrix} \frac{1}{c}(mc^2 + \frac{1}{2}mv^2 + \cdots) \\ m\vec{v} + \cdots \end{pmatrix} \equiv \underbrace{\begin{pmatrix} \frac{1}{c}E_{relativistic} \\ \vec{P}_{relativistic} \end{pmatrix}}_{\text{henceforth we will just call these } E \text{ and } \vec{p}}$

good old 3D momentum

This leads to the famous $E_{rest} = mc^2$, but more importantly ...

Since P^μ is a vector, $P_\mu P^\mu$ is an invariant.

In particular $\underbrace{P_\mu P^\mu}_{\text{Any frame}} = \underbrace{P_{\mu_{rest}} P^{\mu\, rest}}_{\text{Rest frame}}$

$$\Downarrow$$

$$\underbrace{-\frac{E^2}{c^2} + p^2}_{} = -m^2c^2 \quad \text{or} \quad \boxed{E^2 - p^2c^2 = m^2c^4}$$

This "mass-shell" condition relates relativistic energy and momentum to mass, and must be obeyed by all real particles!

Also recall: $P_\mu P^\mu \begin{cases} < 0 & \text{timelike} \Rightarrow m^2 > 0 \quad \text{massive} \\ = 0 & \text{lightlike} \Rightarrow m^2 = 0 \quad \text{massless} \\ > 0 & \text{spacelike} \Rightarrow m^2 < 0 \quad \text{tachyonic} \end{cases}$

To study collisions with P^μ we just define our system to include all colliding particles and then impose: $P^\mu_{tot,i} = P^\mu_{tot,f}$

However there is an incredibly useful trick at our disposal.
If we use $P_i^\mu = P_f^\mu$ then everything (both sides) must be evaluated in a single reference frame (x^μ).
However if we consider:

$$P_{\mu,i} P^\mu_i = P_{\mu,f} P^\mu_f$$

both sides are invariants so we can evaluate them in any frame, even different ones!

$$P_{\mu,i} P^\mu_i = P_{\mu',f} P^{\mu'}_f$$

Note: We would never consider $P_i^\mu = P_f^{\mu'}$!!